A BOOLEAN TASK ALGEBRA FOR REINFORCEMENT LEARNING

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ABSTRACT

We propose a framework for defining a Boolean algebra over the space of tasks. This allows us to formulate new tasks in terms of the negation, disjunction and conjunction of a set of base tasks. We then show that by learning goal-oriented value functions and restricting the transition dynamics of the tasks, an agent can solve these new tasks with no further learning. We prove that by composing these value functions in specific ways, we immediately recover the optimal policies for all tasks expressible under the Boolean algebra.

1 INTRODUCTION

Reinforcement learning (RL) has achieved recent success in a number of difficult, high-dimensional environments (Mnih et al., 2015; Levine et al., 2016; Silver et al., 2017). However, these methods generally require millions of samples from the environment to learn optimal behaviours, limiting their real-world applicability. A major challenge is thus in designing sample-efficient agents that can transfer their existing knowledge to solve new tasks quickly. This is particularly important in a multitask setting, since learning to solve complex tasks from scratch is typically infeasible.

One approach to transfer is *composition* (Todorov, 2009), which allows an agent to leverage existing skills to build complex, novel behaviours. These newly-formed skills can then be used to solve or speed up learning in a new task. In this work, we focus on concurrent composition, where existing base skills are combined to produce new skills (Todorov, 2009; Saxe et al., 2017; Haarnoja et al., 2018; Van Niekerk et al., 2019; Hunt et al., 2019; Peng et al., 2019). This differs from other forms of composition, such as options (Sutton et al., 1999) and hierarchical RL (Bacon et al., 2017), where actions and skills are chained in a temporal sequence.

In this work, we define a Boolean algebra over the space of tasks and optimal value functions. This extends previous composition results to encompass all Boolean operators: conjunction, disjunction, and negation. We then prove that there exists a homomorphism between the task and value function algebras. Given a set of base tasks that have been previously solved by the agent, any new task written as a Boolean expression can immediately be solved without further learning, resulting in a zero-shot super-exponential explosion in the agent's abilities.[1]

2 PRELIMINARIES

We consider tasks modelled by Markov Decision Processes (MDPs). An MDP is defined by the tuple $(\mathcal{S}, \mathcal{A}, \rho, r)$, where (i) $\mathcal{S}$ is the state space, (ii) $\mathcal{A}$ is the action space, (iii) ρ is a Markov transition kernel $(s, a) \mapsto \rho_{(s,a)}$ from $\mathcal{S} \times \mathcal{A}$ to $\mathcal{S}$, and (iv) r is the real-valued reward function bounded by $[r_{\text{MIN}}, r_{\text{MAX}}]$. In this work, we focus on stochastic shortest path problems (Bertsekas & Tsitsiklis, 1991), which model tasks in which an agent must reach some goal. We therefore consider the class of undiscounted MDPs with an absorbing set $\mathcal{G} \subseteq \mathcal{S}$ and whose reward functions differ only in $\mathcal{G}$. For all non-terminal states, we denote the reward $r_{s,a}$ to emphasise that it is constant across tasks.

[1] We provide all relevant proofs and additional experiments in the Appendix

3 BOOLEAN ALGEBRAS FOR TASKS AND VALUE FUNCTIONS

An abstract Boolean algebra is a set $\mathcal{B}$ equipped with operators $\neg, \vee, \wedge$ that satisfy the Boolean axioms of (i) idempotence, (ii) commutativity, (iii) associativity, (iv) absorption, (v) distributivity, (vi) identity, and (vii) complements.[2] Assuming tasks have deterministic transition dynamics and sparse rewards, we can now define a Boolean algebra over a set of tasks.

Theorem 1. *Let $\mathcal{M}$ be a set of tasks. Define $\mathcal{M}_{MAX}, \mathcal{M}_{MIN} \in \mathcal{M}$ to be tasks with the respective reward functions*

$$r_{\mathcal{M}_{MAX}}(s,a) := \begin{cases} r_{MAX}, & \text{if } s \in \mathcal{G} \\ r_{s,a}, & \text{otherwise.} \end{cases} \qquad r_{\mathcal{M}_{MIN}}(s,a) := \begin{cases} r_{MIN}, & \text{if } s \in \mathcal{G} \\ r_{s,a}, & \text{otherwise.} \end{cases}$$

Then $\mathcal{M}$ forms a Boolean algebra with universal bounds $\mathcal{M}_{MIN}$ and $\mathcal{M}_{MAX}$ when equipped with the operators $\neg, \vee, \wedge$ given by:

$$\neg(M) := (\mathcal{S}, \mathcal{A}, \rho, r_{\neg M}), \text{ where } r_{\neg M}(s,a) := (r_{\mathcal{M}_{MAX}}(s,a) + r_{\mathcal{M}_{MIN}}(s,a)) - r_M(s,a)$$

$$\vee(M_1, M_2) := (\mathcal{S}, \mathcal{A}, \rho, r_{M_1 \vee M_2}), \text{ where } r_{M_1 \vee M_2}(s,a) := \max\{r_{M_1}(s,a), r_{M_2}(s,a)\}$$

$$\wedge(M_1, M_2) := (\mathcal{S}, \mathcal{A}, \rho, r_{M_1 \wedge M_2}), \text{ where } r_{M_1 \wedge M_2}(s,a) := \min\{r_{M_1}(s,a), r_{M_2}(s,a)\}$$

Theorem 1 allows us to compose existing tasks together to create new tasks in a principled way.

3.1 EXTENDED VALUE FUNCTIONS

To successfully do zero-shot logical composition we define goal-oriented versions of the reward and value function, given by the following two definitions:

Definition 1. *The extended reward function $\bar{r} : \mathcal{S} \times \mathcal{G} \times \mathcal{A} \to \mathbb{R}$ is given by the mapping*

$$(s,g,a) \mapsto \begin{cases} N & \text{if } g \neq s \in \mathcal{G} \\ r(s,a) & \text{otherwise,} \end{cases} \tag{1}$$

where $N \leq \min\{r_{MIN}, (r_{MIN} - r_{MAX})D\}$, and D is the diameter of the MDP (Jaksch et al., 2010).[3]

To understand why standard value functions are insufficient, consider two tasks that have multiple different goals, but at least one common goal. Clearly, there is a meaningful conjunction between them—namely, achieving the common goal. Now consider an agent that learns standard value functions for both tasks, and which is then required to solve their conjunction without further learning. Note that this is impossible in general, since the regular value function for each task only represents the value of each state with respect to the *nearest* goal. That is, for all states where the nearest goal for each task is *not* the common goal, the agent has no information about that common goal. Conversely, by learning extended value functions, the agent is able to learn the value of achieving all goals, and not simply the nearest one.

Because we require that tasks share the same transition dynamics, we also require that the absorbing set of states is shared. Thus the extended reward function adds the extra constraint that, if the agent enters a terminal state for a *different* task, it should receive the largest penalty possible. In practice, we can simply set N to be the lowest finite value representable by the data type used for rewards.

Definition 2. *The extended Q-value function $\bar{Q} : \mathcal{S} \times \mathcal{G} \times \mathcal{A} \to \mathbb{R}$ is given by the mapping*

$$(s,g,a) \mapsto \bar{r}(s,g,a) + \int_{\mathcal{S}} \bar{V}^{\bar{\pi}}(s',g)\rho_{(s,a)}(ds'), \tag{2}$$

where $\bar{V}^{\bar{\pi}}(s,g) = \mathbb{E}_{\bar{\pi}}\left[\sum_{t=0}^{\infty} \bar{r}(s_t, g, a_t)\right]$. The extended Q-value function is similar to universal value function approximators (UVFAs) (Schaul et al., 2015), but differs in that it uses the extended

[2] We provide a description of these axioms in the Appendix.

[3] The diameter is defined as $D = \max_{s \neq s' \in \mathcal{S}} \min_{\pi} \mathbb{E}\left[T(s'|\pi, s)\right]$, where T is the number of timesteps required to first reach s' from s under π.

reward function definition. It is also similar to DG functions (Kaelbling, 1993), except here we use task-dependent reward functions, as opposed to measuring distance between states.

We can extract the greedy action from the extended value function by first maximising over goals, and then selecting the maximising action: $\pi^*(s) \in \arg\max_{a \in \mathcal{A}} \max_{g \in \mathcal{G}} \bar{Q}^*(s, g, a)$. If we consider the extended value function to be a set of standard value functions (one for each goal), then this is equivalent to first performing generalised policy improvement (Barreto et al., 2017), and then selecting the greedy action.

3.2 A Boolean Algebra for Value Functions

In the same manner we constructed a Boolean algebra over a set of tasks, we can also do so for a set of optimal extended Q-value functions for the corresponding tasks.

Theorem 2. *Let $\bar{\mathcal{Q}}^*$ be the set of optimal extended $\bar{Q}$-value functions for tasks in $\mathcal{M}$. Define $\bar{Q}^*_{MIN}, \bar{Q}^*_{MAX} \in \bar{\mathcal{Q}}^*$ to be the optimal $\bar{Q}$-functions for the tasks $\mathcal{M}_{MIN}, \mathcal{M}_{MAX} \in \mathcal{M}$. Then $\bar{\mathcal{Q}}^*$ forms a Boolean algebra when equipped with the operators $\neg, \vee, \wedge$ given by:*

$$\neg(\bar{Q}^*)(s, g, a) \coloneqq \big(\bar{Q}^*_{MAX}(s, g, a) + \bar{Q}^*_{MIN}(s, g, a)\big) - \bar{Q}^*(s, g, a)$$

$$\vee(\bar{Q}^*_1, \bar{Q}^*_2)(s, g, a) \coloneqq \max\{\bar{Q}^*_1(s, g, a), \bar{Q}^*_2(s, g, a)\}$$

$$\wedge(\bar{Q}^*_1, \bar{Q}^*_2)(s, g, a) \coloneqq \min\{\bar{Q}^*_1(s, g, a), \bar{Q}^*_2(s, g, a)\}$$

3.3 Between Task and Value Function Algebras

Having established a Boolean algebra over tasks and extended value function, we finally state the existence of an equivalence between the two. As a result, if we can write down a task under the Boolean algebra, we can immediately write down the optimal value function for the task.

Theorem 3. *Let $\mathcal{F} : \mathcal{M} \to \bar{\mathcal{Q}}^*$ be any map from $\mathcal{M}$ to $\bar{\mathcal{Q}}^*$ such that $\mathcal{F}(M) = \bar{Q}^*_M$ for all M in $\mathcal{M}$. Then $\mathcal{F}$ is a homomorphism.*

4 Zero-shot Transfer Through Composition

4.1 Learning Base Tasks

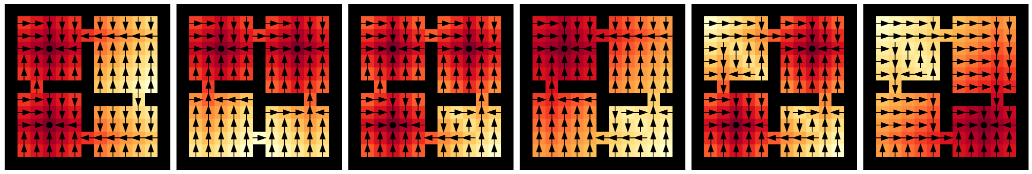
If we know the set of goals (and hence potential base tasks) upfront, then it is easy to select a minimal set of base tasks that can be composed to produce the largest number of composite tasks. We assign a Boolean label to each goal in a table, and then use the columns of the table as base tasks. The goals for each base task are then those goals with value r_{MAX} according to the table. We consider the Four Rooms domain (Sutton et al., 1999). Here the two base tasks we select are M_{T}, which requires that the agent visit either of the top two rooms, and M_{L}, which requires visiting the two left rooms.[4]

4.2 Boolean Composition

Having learned the optimal extended value functions for our base tasks, we can now leverage Theorems 1–3 to solve new tasks with no further learning. Figure 1 illustrates this composition, where an agent is able to immediately solve complex tasks such as exclusive-or. We illustrate a few composite tasks here, but note that in general, if we have K base tasks, then a Boolean algebra allows for 2^{2^K} new tasks to be constructed. Thus having trained on only two tasks, our agent has enough information to solve a total of 16 composite tasks.

By learning extended value functions, an agent can subsequently solve a massive number of tasks; however, the upfront cost of learning is likely to be higher. We investigate the trade-off between the two approaches by investigating how the sample complexity scales with the number of tasks.

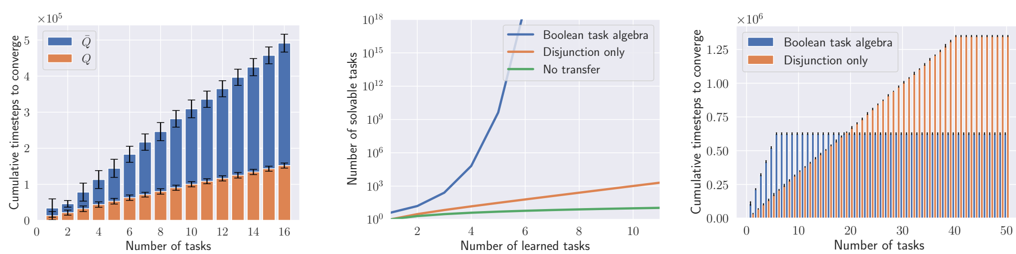
[4]We illustrate this selection procedure in the Appendix.



(a) M_{L} (b) M_{T} (c) $M_{\mathrm{L}} \vee M_{\mathrm{T}}$ (d) $M_{\mathrm{L}} \wedge M_{\mathrm{T}}$ (e) $M_{\mathrm{L}} \veebar M_{\mathrm{T}}$ (f) $M_{\mathrm{L}} \bar{\vee} M_{\mathrm{T}}$

Figure 1: An example of zero-shot Boolean algebraic composition using the learned extended value functions. Arrows represent the optimal action in a given state. (a–b) The learned optimal goal oriented value functions for the base tasks. (c) Zero-shot disjunctive composition. (d) Zero-shot conjunctive composition. (e) Combining operators to model exclusive-or composition. (f) Composition that produces logical nor. Note that the resulting optimal value function can attain a goal not explicitly represented by the base tasks.

We compare to Van Niekerk et al. (2019), who used regular value functions to demonstrate optimal disjunctive composition. We note that while the upfront learning cost is therefore lower, the number of tasks expressible using only disjunction is $2^K - 1$, which is significantly less than the full Boolean algebra. We also test using an extended version of the Four Rooms domain, where additional goals are placed along the sides of all walls, resulting in a total of 40 goals. Empirical results are illustrated by Figure 2. Our results show that while additional samples are needed to learn an extended value function, the agent is able to expand the tasks it can solve super-exponentially. Furthermore, the number of base tasks we need to solve is only logarithmic in the number of goal states. For an environment with K goals, we need to learn only $\lfloor \log_2 K \rfloor + 1$ base tasks, as opposed to the disjunctive approach which requires K base tasks. Thus by sacrificing sample efficiency initially, we achieve an exponential increase in abilities compared to previous work (Van Niekerk et al., 2019).



(a) Cumulative number of samples required to learn optimal extended and regular value functions. Error bars represent standard deviations over 100 runs.

(b) Number of tasks that can be solved as a function of the number of existing tasks solved. Results are plotted on a log-scale.

(c) Cumulative number of samples required to solve tasks in a 40-goal Four Rooms domain. Error bars represent standard deviations over 100 runs.

Figure 2: Results in comparison to the disjunctive composition of Van Niekerk et al. (2019). (a) The number of samples required to learn the extended value function is greater than learning a standard value function. However, both scale linearly and differ only by a constant factor. (b) The extended value functions allow us to solve exponentially more tasks than the disjunctive approach without further learning. (c) In the modified task with 40 goals, we need to learn only 7 base tasks, as opposed to 40 for the disjunctive case.

5 Conclusion

We have shown how to compose tasks using the standard Boolean algebra operators. These composite tasks can be immediately solved by first learning goal-oriented value functions, and then composing them in a similar manner. Our proposed approach is a step towards both interpretable RL—since both the tasks and optimal value functions can be specified using Boolean operators—and the ultimate goal of lifelong learning agents, which are able to solve combinatorially many tasks in a sample-efficient manner.

REFERENCES

P. Bacon, J. Harb, and D. Precup. The option-critic architecture. In *Thirty-First AAAI Conference on Artificial Intelligence*, 2017.

A. Barreto, W. Dabney, R. Munos, J. Hunt, T. Schaul, H. van Hasselt, and D. Silver. Successor features for transfer in reinforcement learning. In *Advances in neural information processing systems*, pp. 4055–4065, 2017.

D.P. Bertsekas and J.N. Tsitsiklis. An analysis of stochastic shortest path problems. *Mathematics of Operations Research*, 16(3):580–595, 1991.

T. Haarnoja, V. Pong, A. Zhou, M. Dalal, P. Abbeel, and S. Levine. Composable deep reinforcement learning for robotic manipulation. *arXiv preprint arXiv:1803.06773*, 2018.

J. Hunt, A. Barreto, T. Lillicrap, and N. Heess. Composing entropic policies using divergence correction. In *Proceedings of the 36th International Conference on Machine Learning*, volume 97 of *Proceedings of Machine Learning Research*, pp. 2911–2920. PMLR, 09–15 Jun 2019.

T. Jaksch, R. Ortner, and P. Auer. Near-optimal regret bounds for reinforcement learning. *Journal of Machine Learning Research*, 11(Apr):1563–1600, 2010.

Leslie Pack Kaelbling. Learning to achieve goals. In *International Joint Conferences on Artificial Intelligence*, pp. 1094–1099, 1993.

S. Levine, C. Finn, T. Darrell, and P. Abbeel. End-to-end training of deep visuomotor policies. *The Journal of Machine Learning Research*, 17(1):1334–1373, 2016.

V. Mnih, K. Kavukcuoglu, D. Silver, A. Rusu, J. Veness, M. Bellemare, A. Graves, M. Riedmiller, A. Fidjeland, G. Ostrovski, et al. Human-level control through deep reinforcement learning. *Nature*, 518(7540):529, 2015.

X. Peng, M. Chang, G. Zhang, P. Abbeel, and S. Levine. MCP: Learning composable hierarchical control with multiplicative compositional policies. *arXiv preprint arXiv:1905.09808*, 2019.

A.M. Saxe, A.C. Earle, and B.S. Rosman. Hierarchy through composition with multitask LMDPs. *Proceedings of the 34th International Conference on Machine Learning*, 70:3017–3026, 2017.

T. Schaul, D. Horgan, K. Gregor, and D. Silver. Universal value function approximators. In *Proceedings of the 32nd International Conference on Machine Learning*, volume 37 of *Proceedings of Machine Learning Research*, pp. 1312–1320, Lille, France, 07–09 Jul 2015. PMLR.

D. Silver, J. Schrittwieser, K. Simonyan, I. Antonoglou, A. Huang, A. Guez, T. Hubert, L. Baker, M. Lai, A. Bolton, et al. Mastering the game of go without human knowledge. *Nature*, 550(7676): 354, 2017.

R. Sutton, D. Precup, and S. Singh. Between MDPs and semi-MDPs: A framework for temporal abstraction in reinforcement learning. *Artificial intelligence*, 112(1-2):181–211, 1999.

E. Todorov. Linearly-solvable Markov decision problems. In *Advances in Neural Information Processing Systems*, pp. 1369–1376, 2007.

E. Todorov. Compositionality of optimal control laws. In *Advances in Neural Information Processing Systems*, pp. 1856–1864, 2009.

B. Van Niekerk, S. James, A. Earle, and B. Rosman. Composing value functions in reinforcement learning. In *Proceedings of the 36th International Conference on Machine Learning*, volume 97 of *Proceedings of Machine Learning Research*, pp. 6401–6409. PMLR, 09–15 Jun 2019.

A APPENDIX

A.1 BOOLEAN ALGEBRA DEFINITION

Definition 3. *A Boolean algebra is a set $\mathcal{B}$ equipped with the binary operators $\vee$ (disjunction) and $\wedge$ (conjunction), and the unary operator $\neg$ (negation), which satisfies the following Boolean algebra axioms for a, b, c in $\mathcal{B}$:*

(i) Idempotence: $a \wedge a = a \vee a = a$.

(ii) Commutativity: $a \wedge b = b \wedge a$ *and* $a \vee b = b \vee a$.

(iii) Associativity: $a \wedge (b \wedge c) = (a \wedge b) \wedge c$ *and* $a \wedge (b \vee c) = (a \vee b) \vee c$.

(iv) Absorption: $a \wedge (a \vee b) = a \vee (a \wedge b) = a$.

(v) Distributivity: $a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$ *and* $a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)$.

(vi) Identity: there exists $\mathbf{0}, \mathbf{1}$ *in* $\mathcal{B}$ *such that*

$$
\begin{aligned}
\mathbf{0} \wedge a &= \mathbf{0} \\
\mathbf{0} \vee a &= a \\
\mathbf{1} \wedge a &= a \\
\mathbf{1} \vee a &= \mathbf{1}
\end{aligned}
$$

(vii) Complements: for every a *in* $\mathcal{B}$*, there exists an element* a' *in* $\mathcal{B}$ *such that* $a \wedge a' = \mathbf{0}$ *and* $a \vee a' = \mathbf{1}$.

A.2 ASSUMPTIONS

We consider a family of related MDPs $\mathcal{M}$ restricted by the following assumptions:

Assumption 1. *For all tasks in a set of tasks $\mathcal{M}$, (i) the tasks share the same state space, action space and transition dynamics, (ii) the transition dynamics are deterministic, and (iii) reward functions between tasks differ only on the absorbing set $\mathcal{G}$.*

Assumption 2. *For all tasks in a set of tasks $\mathcal{M}$ which adhere to Assumption 1, the set of possible terminal rewards consists of only two values. That is, for all (g, a) in $\mathcal{G} \times \mathcal{A}$, we have that $r(g, a) \in \{r_{MIN}, r_{MAX}\} \subset \mathbb{R}$ with $r_{MIN} \leq r_{MAX}$. For all non-terminal states, we denote the reward $r_{s,a}$ to emphasise that it is constant across tasks.*

Assumption 1 is similar to that of Todorov (2007) and identical to Van Niekerk et al. (2019), and imply that each task can be uniquely specified by its reward function. Furthermore, we note that Assumption 2 is only necessary to formally define the Boolean algebra. Our proof for zero-shot composition still holds without it.[5] Although we have placed restrictions on the reward functions, the above formulation still allows for a large number of tasks to be represented. Importantly, sparse rewards can be formulated under these restrictions.

[5]See the proof for theorem 3 in Appendix A.6

A.3 Proof for Theorem 1

Theorem 1. *Let $\mathcal{M}$ be a set of tasks. Define $\mathcal{M}_{MAX}, \mathcal{M}_{MIN} \in \mathcal{M}$ to be tasks with the respective reward functions*

$$r_{\mathcal{M}_{MAX}} : \mathcal{S} \times \mathcal{A} \to \mathbb{R} \qquad (s,a) \mapsto \begin{cases} r_{MAX}, & \text{if } s \in \mathcal{G} \\ r_{s,a}, & \text{otherwise.} \end{cases}$$

$$r_{\mathcal{M}_{MIN}} : \mathcal{S} \times \mathcal{A} \to \mathbb{R} \qquad (s,a) \mapsto \begin{cases} r_{MIN}, & \text{if } s \in \mathcal{G} \\ r_{s,a}, & \text{otherwise.} \end{cases}$$

Then $\mathcal{M}$ forms a Boolean algebra with universal bounds $\mathcal{M}_{MIN}$ and $\mathcal{M}_{MAX}$ when equipped with the following operators:

$$\begin{aligned} \neg : \; & \mathcal{M} \to \mathcal{M} \\ & M \mapsto (\mathcal{S}, \mathcal{A}, \rho, r_{\neg M}), \text{ where } r_{\neg M} : \mathcal{S} \times \mathcal{A} \to \mathbb{R} \\ & \qquad (s,a) \mapsto (r_{\mathcal{M}_{MAX}}(s,a) + r_{\mathcal{M}_{MIN}}(s,a)) - r_M(s,a) \end{aligned}$$

$$\begin{aligned} \vee : \; & \mathcal{M} \times \mathcal{M} \to \mathcal{M} \\ & (M_1, M_2) \mapsto (\mathcal{S}, \mathcal{A}, \rho, r_{M_1 \vee M_2}), \text{ where } r_{M_1 \vee M_2} : \mathcal{S} \times \mathcal{A} \to \mathbb{R} \\ & \qquad (s,a) \mapsto \max\{r_{M_1}(s,a), r_{M_2}(s,a)\} \end{aligned}$$

$$\begin{aligned} \wedge : \; & \mathcal{M} \times \mathcal{M} \to \mathcal{M} \\ & (M_1, M_2) \mapsto (\mathcal{S}, \mathcal{A}, \rho, r_{M_1 \wedge M_2}), \text{ where } r_{M_1 \wedge M_2} : \mathcal{S} \times \mathcal{A} \to \mathbb{R} \\ & \qquad (s,a) \mapsto \min\{r_{M_1}(s,a), r_{M_2}(s,a)\} \end{aligned}$$

Proof. Let $M_1, M_2 \in \mathcal{M}$. We show that $\neg, \vee, \wedge$ satisfy the Boolean properties (i) – (vii).

(i)–(v): These easily follow from the fact that the $\min$ and $\max$ functions satisfy the idempotent, commutative, associative, absorption and distributive laws.

(vi): Let $r_{\mathcal{M}_{MAX} \wedge M_1}$ and r_{M_1} be the reward functions for $\mathcal{M}_{MAX} \wedge M_1$ and M_1 respectively. Then for all (s,a) in $\mathcal{S} \times \mathcal{A}$,

$$\begin{aligned} r_{\mathcal{M}_{MAX} \wedge M_1}(s,a) &= \begin{cases} \min\{r_{MAX}, r_{M_1}(s,a)\}, & \text{if } s \in \mathcal{G} \\ \min\{r_{s,a}, r_{s,a}\}, & \text{otherwise.} \end{cases} \\ &= \begin{cases} r_{M_1}(s,a), & \text{if } s \in \mathcal{G} \\ r_{s,a}, & \text{otherwise.} \end{cases} \\ &= r_{M_1}(s,a). \end{aligned}$$

Thus $\mathcal{M}_{MAX} \wedge M_1 = M_1$. Similarly $\mathcal{M}_{MAX} \vee M_1 = \mathcal{M}_{MAX}$, $\mathcal{M}_{MIN} \wedge M_1 = \mathcal{M}_{MIN}$, and $\mathcal{M}_{MIN} \vee M_1 = M_1$. Hence $\mathcal{M}_{MIN}$ and $\mathcal{M}_{MAX}$ are the universal bounds of $\mathcal{M}$.

(vii): Let $r_{M_1 \wedge \neg M_1}$ be the reward function for $M_1 \wedge \neg M_1$. Then for all (s,a) in $\mathcal{S} \times \mathcal{A}$,

$$\begin{aligned} r_{M_1 \wedge \neg M_1}(s,a) &= \begin{cases} \min\{r_{M_1}(s,a), (r_{MAX} + r_{MIN}) - r_{M_1}(s,a)\}, & \text{if } s \in \mathcal{G} \\ \min\{r_{s,a}, (r_{s,a} + r_{s,a}) - r_{s,a}\}, & \text{otherwise.} \end{cases} \\ &= \begin{cases} r_{MIN}, & \text{if } s \in \mathcal{G} \text{ and } r_{M_1}(s,a) = r_{MAX} \\ r_{MIN}, & \text{if } s \in \mathcal{G} \text{ and } r_{M_1}(s,a) = r_{MIN} \\ r_{s,a}, & \text{otherwise.} \end{cases} \\ &= r_{\mathcal{M}_{MIN}}(s,a). \end{aligned}$$

Thus $M_1 \wedge \neg M_1 = \mathcal{M}_{MIN}$, and similarly $M_1 \vee \neg M_1 = \mathcal{M}_{MAX}$.

□

A.4 Lemmas and Corollaries

The standard reward functions and value functions can be recovered from their extended versions through the following lemma.

Lemma 1. *Let $r_M, \bar{r}_M, Q^*_M, \bar{Q}^*_M$ be the reward function, extended reward function, optimal Q-value function, and optimal extended Q-value function for a task M in $\mathcal{M}$. Then for all (s,a) in $\mathcal{S} \times \mathcal{A}$, we have (i) $r_M(s,a) = \max_{g \in \mathcal{G}} \bar{r}_M(s,g,a)$, and (ii) $Q^*_M(s,a) = \max_{g \in \mathcal{G}} \bar{Q}^*_M(s,g,a)$.*

Proof.

(i):

$$
\begin{aligned}
\max_{g \in \mathcal{G}} \bar{r}_M(s,g,a) &= \begin{cases} \max\{N, r_M(s,a)\}, & \text{if } s \in \mathcal{G} \\ \max_{g \in \mathcal{G}} r_M(s,a), & \text{otherwise.} \end{cases} \\
&= r_M(s,a) \qquad\qquad (N \leq r_{\text{MIN}} \leq r_M(s,a) \text{ by definition}).
\end{aligned}
$$

(ii): Each g in $\mathcal{G}$ can be thought of as defining an MDP $M_g := (\mathcal{S}, \mathcal{A}, \rho, r_{M_g})$ with reward function $r_{M_g}(s,a) := \bar{r}_M(s,g,a)$ and optimal Q-value function $Q^*_{M_g}(s,a) = \bar{Q}^*_M(s,g,a)$. Then using (i) we have $r_M(s,a) = \max_{g \in \mathcal{G}} r_{M_g}(s,a)$ and from Van Niekerk et al. (2019, Corollary 1), we have that $Q^*_M(s,a) = \max_{g \in \mathcal{G}} Q^*_{M_g}(s,a) = \max_{g \in \mathcal{G}} \bar{Q}^*_M(s,g,a)$.

□

In the same way, we can also recover the optimal policy from these extended value functions by first applying Lemma 1, and acting greedily with respect to the resulting value function.

Lemma 2. *Denote $\mathcal{S}^- = \mathcal{S} \setminus \mathcal{G}$ as the non-terminal states of $\mathcal{M}$. Let $M_1, M_2 \in \mathcal{M}$, and let each g in $\mathcal{G}$ define MDPs $M_{1,g}$ and $M_{2,g}$ with reward functions*

$$r_{M_{1,g}} := \bar{r}_{M_1}(s,g,a) \textit{ and } r_{M_{2,g}} := \bar{r}_{M_2}(s,g,a) \textit{ for all } (s,a) \textit{ in } \mathcal{S} \times \mathcal{A}.$$

Then for all g in $\mathcal{G}$ and s in $\mathcal{S}^-$,

$$\pi^*_g(s) \in \arg\max_{a \in \mathcal{A}} Q^*_{M_{1,g}}(s,a) \textit{ iff } \pi^*_g(s) \in \arg\max_{a \in \mathcal{A}} Q^*_{M_{2,g}}(s,a).$$

Proof. Let $g \in \mathcal{G}, s \in \mathcal{S}^-$ and let π^*_g be defined by

$$\pi^*_g(s') \in \arg\max_{a \in \mathcal{A}} Q^*_{M_1,g}(s,a) \text{ for all } s' \in \mathcal{S}.$$

If g is unreachable from s, then we are done since for all (s',a) in $\mathcal{S} \times \mathcal{A}$ we have

$$
\begin{aligned}
g \neq s' &\implies r_{M_{1,g}}(s',a) = \begin{cases} N, & \text{if } s' \in \mathcal{G} \\ r_{s',a}, & \text{otherwise} \end{cases} = r_{M_{2,g}}(s',a) \\
&\implies M_{1,g} = M_{2,g}.
\end{aligned}
$$

If g *is* reachable from s, then we show that following π^*_g must reach g. Since π^*_g is proper, it must reach a terminal state $g' \in \mathcal{G}$. Assume $g' \neq g$. Let π_g be a policy that produces the shortest trajectory to g. Let $G^{\pi^*_g}$ and G^{π_g} be the returns for the respective policies. Then,

$$
\begin{aligned}
& G^{\pi^*_g} \geq G^{\pi_g} \\
& \implies G^{\pi^*_g}_{T-1} + r_{M_{1,g}}(g', \pi^*_g(g')) \geq G^{\pi_g}, \\
& \text{where } G^{\pi^*_g}_{T-1} = \sum_{t=0}^{T-1} r_{M_{1,g}}(s_t, \pi^*_g(s_t)) \text{ and } T \text{ is the time at which } g' \text{ is reached.} \\
& \implies G^{\pi^*_g}_{T-1} + N \geq G^{\pi_g}, \text{ since } g \neq g' \in \mathcal{G} \\
& \implies N \geq G^{\pi_g} - G^{\pi^*_g}_{T-1} \\
& \implies (r_{\text{MIN}} - r_{\text{MAX}})D \geq G^{\pi_g} - G^{\pi^*_g}_{T-1}, \text{ by definition of } N \\
& \implies G^{\pi^*_g}_{T-1} - r_{\text{MAX}}D \geq G^{\pi_g} - r_{\text{MIN}}D, \text{ since } G^{\pi_g} \geq r_{\text{MIN}}D \\
& \implies G^{\pi^*_g}_{T-1} - r_{\text{MAX}}D \geq 0 \\
& \implies G^{\pi^*_g}_{T-1} \geq r_{\text{MAX}}D.
\end{aligned}
$$

But this is a contradiction since the result obtained by following an optimal trajectory up to a terminal state without the reward for entering the terminal state must be strictly less that receiving r_{MAX} for every step of the longest possible optimal trajectory. Hence we must have $g' = g$. Similarly, all optimal policies of $M_{2,g}$ must reach g. Hence $\pi^*_g(s) \in \arg\max_{a \in \mathcal{A}} Q^*_{M_{2,g}}(s, a)$. Since M_1 and M_2 are arbitrary elements of $\mathcal{M}$, the reverse implication holds too.

□

Finally, much like the regular definition of value functions, the extended Q-value function can be written as the sum of rewards received by the agent until first encountering a terminal state.

Corollary 1. *Denote $G^*_{s:g,a}$ as the sum of rewards starting from s and taking action a up until, but not including, g. Then let $M \in \mathcal{M}$ and $\bar{Q}^*_M$ be the extended Q-value function. Then for all $s \in \mathcal{S}, g \in \mathcal{G}, a \in \mathcal{A}$, there exists a $G^*_{s:g,a} \in \mathbb{R}$ such that*

$$\bar{Q}^*_M(s, g, a) = G^*_{s:g,a} + \bar{r}_M(s', g, a'), \text{ where } s' \in \mathcal{G} \text{ and } a' = \arg\max_{b \in \mathcal{A}} \bar{r}_M(s', g, b).$$

Proof. This follows directly from Lemma 2. Since all tasks $M \in \mathcal{M}$ share the same optimal policy π^*_g up to (but not including) the goal state $g \in \mathcal{G}$, their return $G^{\pi^*_g}_{T-1} = \sum_{t=0}^{T-1} r_M(s_t, \pi^*_g(s_t))$ is the same up to (but not including) g. □

A.5 Proof for Theorem 2

Theorem 2. *Let $\bar{\mathcal{Q}}^*$ be the set of optimal extended $\bar{Q}$-value functions for tasks in $\mathcal{M}$. Define $\bar{Q}^*_{MIN}, \bar{Q}^*_{MAX} \in \bar{\mathcal{Q}}^*$ to be the optimal $\bar{Q}$-functions for the tasks $\mathcal{M}_{MIN}, \mathcal{M}_{MAX} \in \mathcal{M}$. Then $\bar{\mathcal{Q}}^*$ forms a Boolean algebra when equipped with the following operators:*

$$
\begin{aligned}
\neg :\ & \bar{\mathcal{Q}}^* \to \bar{\mathcal{Q}}^* \\
& \bar{Q}^* \mapsto \neg \bar{Q}^*, \text{ where } \neg \bar{Q}^* : \mathcal{S} \times \mathcal{G} \times \mathcal{A} \to \mathbb{R} \\
& \qquad (s, g, a) \mapsto \left(\bar{Q}^*_{MAX}(s, g, a) + \bar{Q}^*_{MIN}(s, g, a)\right) - \bar{Q}^*(s, g, a)
\end{aligned}
$$

$$
\begin{aligned}
\vee :\ & \bar{\mathcal{Q}}^* \times \bar{\mathcal{Q}}^* \to \bar{\mathcal{Q}}^* \\
& (\bar{Q}^*_1, \bar{Q}^*_2) \mapsto \bar{Q}^*_1 \vee \bar{Q}^*_2, \text{ where } \bar{Q}^*_1 \vee \bar{Q}^*_2 : \mathcal{S} \times \mathcal{G} \times \mathcal{A} \to \mathbb{R} \\
& \qquad (s, g, a) \mapsto \max\{\bar{Q}^*_1(s, g, a), \bar{Q}^*_2(s, a)\}
\end{aligned}
$$

$$\begin{aligned}
\wedge: \ & \bar{\mathcal{Q}}^* \times \bar{\mathcal{Q}}^* \to \bar{\mathcal{Q}}^* \\
& (\bar{Q}_1^*, \bar{Q}_2^*) \mapsto \bar{Q}_1^* \wedge \bar{Q}_2^*, \ \textit{where } \bar{Q}_1^* \wedge \bar{Q}_2^* : \ \mathcal{S} \times \mathcal{G} \times \mathcal{A} \to \mathbb{R} \\
& \qquad\qquad\qquad\qquad\qquad\qquad\qquad (s, g, a) \mapsto \min\{\bar{Q}_1^*(s, g, a), \bar{Q}_2^*(s, a)\}
\end{aligned}$$

Proof. Let $\bar{Q}^*_{M_1}, \bar{Q}^*_{M_2} \in \bar{\mathcal{Q}}^*$ be the optimal $\bar{Q}$-value functions for tasks $M_1, M_2 \in \mathcal{M}$ with reward functions r_{M_1} and r_{M_2}. We show that $\neg, \vee, \wedge$ satisfy the Boolean properties (i) – (vii).

(i)–(v): These follow directly from the properties of the $\min$ and $\max$ functions.

(vi): For all (s, g, a) in $\mathcal{S} \times \mathcal{G} \times \mathcal{A}$,

$$\begin{aligned}
(\bar{Q}^*_{MAX} \wedge \bar{Q}^*_{M_1})(s, g, a) &= \min\{(\bar{Q}^*_{MAX}(s, g, a), \bar{Q}^*_{M_1}(s, g, a)\} \\
&= \min\{G^*_{s:g,a} + \bar{r}_{\mathcal{M}_{MAX}}(s', g, a'), G^*_{s:g,a} + \bar{r}_{M_1}(s', g, a')\} \\
&\quad \text{(from Corollary 1)} \\
&= G^*_{s:g,a} + \min\{\bar{r}_{\mathcal{M}_{MAX}}(s', g, a'), \bar{r}_{M_1}(s', g, a')\} \\
&= G^*_{s:g,a} + \bar{r}_{M_1}(s', g, a') \\
&= \bar{Q}^*_{M_1}(s, g, a).
\end{aligned}$$

Similarly, $\bar{Q}^*_{MAX} \vee \bar{Q}^*_{M_1} = \bar{Q}^*_{MAX}$, $\bar{Q}^*_{MIN} \wedge \bar{Q}^*_{M_1} = \bar{Q}^*_{MIN}$, and $\bar{Q}^*_{MIN} \vee \bar{Q}^*_{M_1} = \bar{Q}^*_{M_1}$.

(vii): For all (s, g, a) in $\mathcal{S} \times \mathcal{G} \times \mathcal{A}$,

$$\begin{aligned}
(\bar{Q}^*_{M_1} \wedge \neg\bar{Q}^*_{M_1})(s, g, a) &= \min\{\bar{Q}^*_{M_1}(s, g, a), (\bar{Q}^*_{MAX}(s, g, a) - \bar{Q}^*_{MIN}(s, g, a)) \\
&\quad - \bar{Q}^*_{M_1}(s, g, a)\} \\
&= G^*_{s:g,a} + \min\{\bar{r}_{M_1}(s', g, a'), (\bar{r}_{\mathcal{M}_{MAX}}(s', g, a') \\
&\quad + \bar{r}_{\mathcal{M}_{MIN}}(s', g, a')) - \bar{r}_{M_1}(s', g, a')\} \\
&= G^*_{s:g,a} + \bar{r}_{\mathcal{M}_{MIN}}(s', g, a') \\
&= \bar{Q}^*_{MIN}(s, g, a).
\end{aligned}$$

Similarly, $\bar{Q}^*_{M_1} \vee \neg\bar{Q}^*_{M_1} = \bar{Q}^*_{MAX}$.

$\square$

A.6 Proof for Theorem 3

Theorem 3. *Let $\mathcal{F} : \mathcal{M} \to \bar{\mathcal{Q}}^*$ be any map from $\mathcal{M}$ to $\bar{\mathcal{Q}}^*$ such that $\mathcal{F}(M) = \bar{Q}^*_M$ for all M in $\mathcal{M}$. Then $\mathcal{F}$ is a homomorphism.*

Proof. Let $M_1, M_2 \in \mathcal{M}$. Then for all (s, g, a) in $\mathcal{S} \times \mathcal{G} \times \mathcal{A}$,

$$\begin{aligned}
\bar{Q}^*_{\neg M_1}(s, g, a) &= G^*_{s:g,a} + \bar{r}_{\neg M_1}(s', g, a') \quad \text{(from Corollary 1)} \\
&= G^*_{s:g,a} + (\bar{r}_{\mathcal{M}_{MAX}}(s', g, a') + \bar{r}_{\mathcal{M}_{MIN}}(s', g, a')) - \bar{r}_{M_1}(s', g, a') \\
&= \big[(G^*_{s:g,a} + \bar{r}_{\mathcal{M}_{MAX}}(s', g, a')) + (G^*_{s:g,a} + \bar{r}_{\mathcal{M}_{MIN}}(s', g, a'))\big] \\
&\quad - (G^*_{s:g,a} + \bar{r}_{M_1}(s', g, a')) \\
&= \big[\bar{Q}^*_{MAX}(s, g, a) + \bar{Q}^*_{MIN}(s, g, a)\big] - \bar{Q}^*_{M_1}(s, g, a) \\
&= \neg\bar{Q}^*_{M_1}(s, g, a) \\
\implies \mathcal{F}(\neg M_1) &= \neg\mathcal{F}(M_1)
\end{aligned}$$

$$
\begin{aligned}
\bar{Q}^*_{M_1 \vee M_2}(s,g,a) &= G^*_{s:g,a} + \bar{r}_{M_1 \vee M_2}(s',g,a') \\
&= G^*_{s:g,a} + \max\{\bar{r}_{M_1}(s',g,a'), \bar{r}_{M_2}(s',g,a'')\} \\
&= \max\{G^*_{s:g,a} + \bar{r}_{M_1}(s',g,a'), G^*_{s:g,a} + \bar{r}_{M_2}(s',g,a'')\} \\
&= \max\{\bar{Q}^*_{M_1}(s,g,a), \bar{Q}^*_{M_2}(s,g,a)\} \\
&= (\bar{Q}^*_{M_1} \vee \bar{Q}^*_{M_2})(s,g,a) \\
\implies \mathcal{F}(M_1 \vee M_2) &= \mathcal{F}(M_1) \vee \mathcal{F}(M_2).
\end{aligned}
$$

Similarly $\mathcal{F}(M_1 \wedge M_2) = \mathcal{F}(M_1) \wedge \mathcal{F}(M_2)$. □

A.7 Goal-oriented Q-learning

Below we list the pseudocode for the modified Q-learning algorithm used in the four-rooms domain.

Algorithm 1: Goal-oriented Q-learning

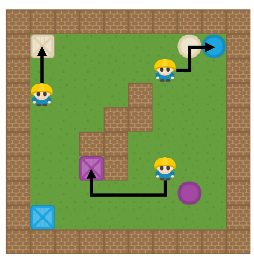
```
Input: Learning rate α, discount factor γ, exploration constant ε, lower-bound return N
Initialise Q : S × S × A → ℝ arbitrarily
G ← ∅
while Q is not converged do
    Initialise state s
    while s is not terminal do
        if G = ∅ then
            Select random action a
        else
            a ← { argmax_{b∈A} ( max_{t∈G} Q(s, t, b) )   with probability 1 − ε
                { a random action                          with probability ε
        end
        Choose a from s according to policy derived from Q
        Take action a, observe r and s′
        foreach g ∈ G do
            if s′ is terminal then
                if s′ ≠ g then
                    δ ← N
                else
                    δ ← r − Q(s, g, a)
                end
            else
                δ ← r + γ max_b Q(s′, g, b) − Q(s, g, a)
            end
            Q(s, g, a) ← Q(s, g, a) + αδ
        end
        s ← s′
    end
    G ← G ∪ {s}
end
return Q
```

Figure 3: A Q-learning algorithm for learning extended value functions. Note that the greedy action selection step is equivalent to generalised policy improvement (Barreto et al., 2017) over the set of extended value functions.

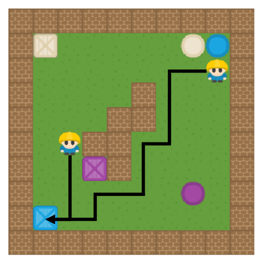
A.8 Composition with Function Approximation

Here we demonstrate that our compositional approach can also be used to tackle high-dimensional domains where function approximation is required. We use the same video game environment as Van Niekerk et al. (2019), where an agent must navigate a 2D world and collect objects of different shapes and colours. The state space is an 84×84 RGB image, and the agent is able to move in any of the four cardinal directions. The agent also possesses a `pick-up` action, which allows it to collect an object when standing on top of it. There are two shapes (squares and circles) and three colours (blue, beige and purple) for a total of six unique objects. The position of the agent is randomised at the start of each episode.

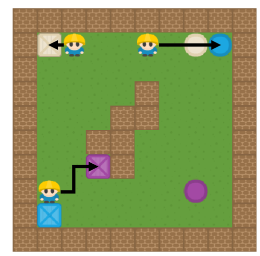
We modify deep Q-learning (Mnih et al., 2015) to learn extended action-value functions.[6] Our approach differs in that the network takes a goal state as additional input (again specified as an RGB image). Additionally, when a terminal state is encountered, it is added to the collection of goals seen so far, and when learning updates occur, these goals are sampled randomly from a replay buffer. We first learn to solve two base tasks: collecting blue objects, and collecting squares, which can then be composed to solve new tasks immediately.



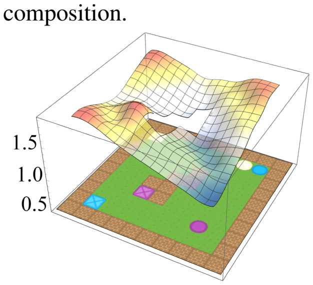
(a) Trajectories for disjunctive composition.



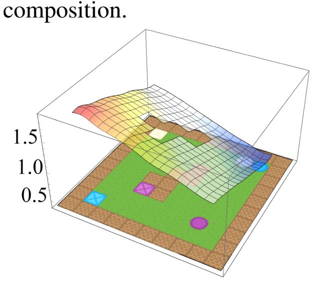
(b) Trajectories for conjunctive composition.



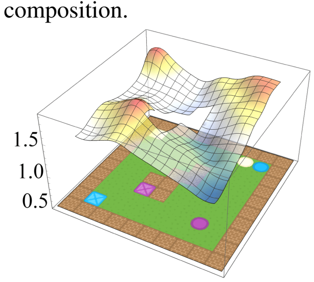
(c) Trajectories for exclusive-or composition.



(d) Value function for disjunctive composition.



(e) Value function for conjunctive composition.



(f) Value function for exclusive-or composition.

Figure 4: By composing extended value functions from the base tasks (collecting blue objects, and collecting squares), we can act optimally in new tasks with no further learning. To generate the value functions, we place the agent at every location and compute the maximum output of the network over all goals and actions. We then interpolate between the points to smooth the graph. Any error in the visualisation is due to the use of non-linear function approximation.

We demonstrate composition characterised by (i) disjunction, (ii) conjunction and (iii) exclusive-or. This corresponds to tasks where the target items are: (i) blue or square, (ii) blue squares, and (iii) blue or squares, but not blue squares. Figure 4 illustrates sample trajectories, as well as the subsequent composed value functions, for the respective tasks.

[6]The hyperparameters and network architecture are listed in the Appendix

A.9 Investigating Practical Considerations

The theoretical results presented in this work rely on Assumptions 1 and 2, which restrict the tasks' transition dynamics and reward functions in potentially problematic ways. Although this is necessary to prove that Boolean algebraic composition results in optimal value functions, in this section we investigate whether these can be practically ignored. In particular, we investigate two restrictions: the requirement that tasks share the same terminal states, and the impact of using dense rewards.

A.9.1 Four Rooms Experiments

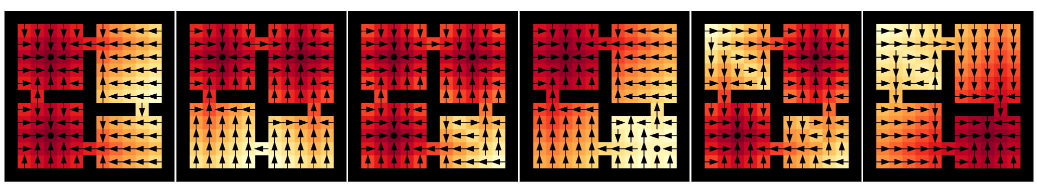
We use the same setup as the experiment outlined in Section 4, but modify it in two ways. We first investigate the difference between using sparse and dense rewards. Our sparse reward function is defined as

$$r_{\text{sparse}}(s,a,s') = \begin{cases} 2 & \text{if } s' \in \mathcal{G} \\ -0.1 & \text{otherwise,} \end{cases}$$

and we use a dense reward function similar to Peng et al. (2019):

$$r_{\text{dense}}(s,a,s') = \frac{0.1}{|\mathcal{G}|}\sum_{g\in\mathcal{G}} \exp(\frac{|s'-g|^2}{4}) + r_{\text{sparse}}(s,a,s')$$

Using this dense reward function, we again learn to solve the two base task M_T (reaching the centre of the top two rooms) and M_L (reaching the centre of the left two rooms). We then compose them to solve a variety of tasks, with the resulting value functions illustrated by Figure 5.



(a) M_{L} (b) M_{T} (c) $M_{\mathrm{L}} \vee M_{\mathrm{T}}$ (d) $M_{\mathrm{L}} \wedge M_{\mathrm{T}}$ (e) $M_{\mathrm{L}} \veebar M_{\mathrm{T}}$ (f) $M_{\mathrm{L}} \bar{\vee} M_{\mathrm{T}}$

Figure 5: An example of Boolean algebraic composition using the learned extended value functions with dense rewards. Arrows represent the optimal action in a given state. (a–b) The learned optimal goal oriented value functions for the base tasks with dense rewards. (c) Disjunctive composition. (d) Conjunctive composition. (e) Combining operators to model exclusive-or composition. (f) Composition that produces logical nor. We note that the resulting value functions are very similar to those produced in the sparse reward setting.

We also modify the domain so that tasks need not share the same terminating states (that is, if the agent enters a terminating state for a *different* task, the episode does not terminate and the agent can continue as if it were a normal state). This results in four versions of the experiment:

(i) `sparse reward, same absorbing set`
(ii) `sparse reward, different absorbing set`
(iii) `dense reward, same absorbing set`
(iv) `dense reward, different absorbing set`

We learn extended value functions for each of the above setups, and then compose them to solve each of the 2^4 tasks representable in the Boolean algebra. We measure each composed value functions by evaluating its policy in the sparse reward setting, averaging results over 100000 episodes. The results are given by Figure 6.

Our results indicate that extended value functions learned in the theoretically optimal manner (`sparse reward, same absorbing set`) are indeed optimal. However, for the majority of the tasks, relaxing the restrictions on terminal states and reward functions results in policies that are either identical or very close to optimal.

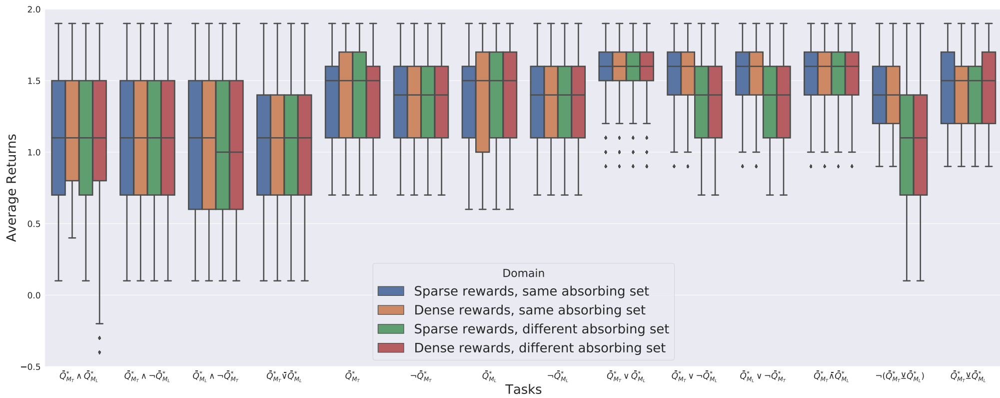
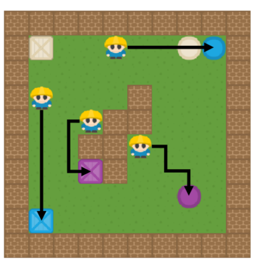


Figure 6: Box plots indicating returns for each of the 16 compositional tasks, and for each of the four variations of the domain. Results are collected over 100000 episodes with random start positions.

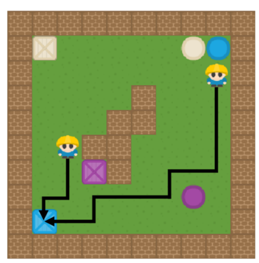
A.9.2 Function Approximation Experiments

In this section we investigate whether we can again loosen some of the restrictive assumptions when tackling high-dimensional environments. In particular, we run the same experiments as those presented in Section A.8, but modify the domain so that (i) tasks need not share the same absorbing set, (ii) the `pickup-up` action is removed (the agent immediately collects an object when reaching it), and (iii) the position of every object is randomised at the start of each episode.

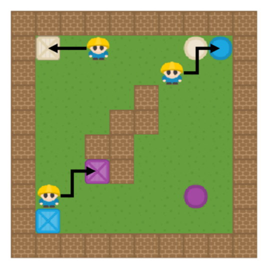
We first learn to solve three base tasks: collecting blue objects, collecting purple objects, and collecting squares , which can then be composed to solve new tasks immediately. We then demonstrate composition characterised by disjunction, conjunction and exclusive-or, with the resulting trajectories and value functions illustrated by Figure 7.



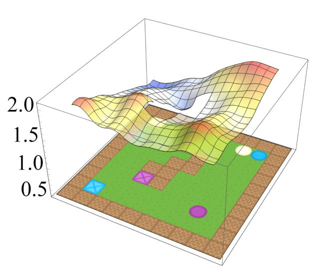
(a) Trajectories for disjunctive composition (collect blue or purple objects).



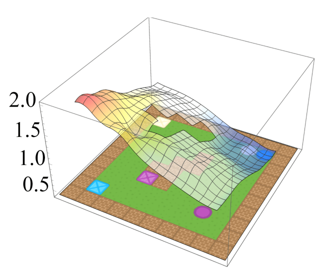
(b) Trajectories for conjunctive composition (collect blue squares).



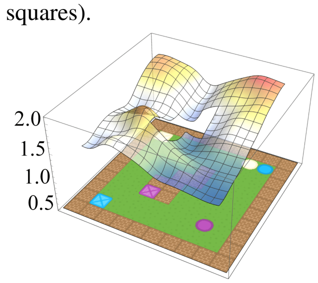
(c) Trajectories for exclusive-or composition (collect blue or square objects, but not blue squares).



(d) Value function for disjunctive composition.



(e) Value function for conjunctive composition.



(f) Value function for exclusive-or composition.

Figure 7: Results for the video game environment with relaxed assumptions. We generate value functions to solve the disjunction of blue and purple tasks, and the conjunction and exclusive-or of blue and square tasks.

In summary, we have shown that our compositional approach offers strong empirical performance, even when the theoretical assumptions are violated. Finally, we expect that, in general, the errors due to these violations will be far outweighed by the errors due to non-linear function approximation.

A.10 Selecting Base Tasks

The Four Rooms domain requires the agent to navigate to one of the centres of the rooms in the environment. Figure 8 illustrates the layout of the environment and the goals the agent must reach.

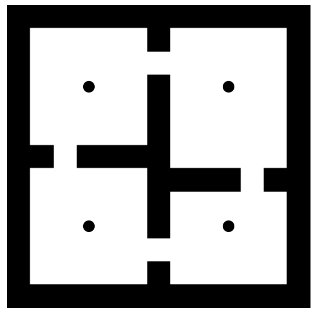


Figure 8: The layout of the Four Rooms domain. The circles indicate goals the agent must reach. We will refer to the goals as `top-left`, `top-right`, `bottom-left`, and `bottom-right`.

Since we know the goals upfront, we can select a minimal set of base tasks by assigning each goal a Boolean number, and then using the columns of the table to select the tasks. To illustrate, we assign Boolean numbers to the goals as follows:

x_1	x_2	Goals
r_{MIN}	r_{MIN}	`bottom-right`
r_{MIN}	r_{MAX}	`bottom-left`
r_{MAX}	r_{MIN}	`top-right`
r_{MAX}	r_{MAX}	`top-left`

Table 1: Assigning labels to the individual goals. The two Boolean variables, x_1 and x_2, represent the goals for the base tasks the agent will train on.

As there are four goals, we can represent each uniquely with just two Boolean variables. Each column in Table 1 represents a base task, where the set of goals for each task are those goals assigned a value r_{MAX}. We thus have two base tasks corresponding to $x_1 = \{\texttt{top-right}, \texttt{top-left}\}$ and $x_2 = \{\texttt{bottom-left}, \texttt{top-left}\}$.

A.11 DQN Architecture and Hyperparameters

In our experiments, we used a DQN with the following architecture:

1. Three convolutional layers:
   (a) Layer 1 has 6 input channels, 32 output channels, a kernel size of 8 and a stride of 4.
   (b) Layer 2 has 32 input channels, 64 output channels, a kernel size of 4 and a stride of 2.
   (c) Layer 3 has 64 input channels, 64 output channels, a kernel size of 3 and a stride of 1.
2. Two fully-connected linear layers:
   (a) Layer 1 has input size 3136 and output size 512 and uses a ReLU activation function.
   (b) Layer 2 has input size 512 and output size 4 with no activation function.

We used the ADAM optimiser with batch size 32 and a learning rate of 10^{-4}. We trained every 4 timesteps and update the target Q-network every 1000 steps. Finally, we used ϵ-greedy exploration, annealing ϵ to 0.01 over 100000 timesteps.